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The Gromov–Hausdorff distance between compact metric spaces

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Resum *(CAT)*
Aquest treball proporciona una introducció a la distància de Gromov–Hausdorff, discutim la seva definició original i la seva relació amb les correspondències entre espais. Demostrem que la distància de Gromov–Hausdorff serveix com a mètrica per al conjunt de classes d'isometria d'espais mètrics compactes. Els objectius principals d'aquest estudi són establir l'existència d'una pseudomètrica en la unió disjunta de X amb Y que aconsegueix la distància de Gromov–Hausdorff entre espais compactes X i Y, i per establir límits per al Gromov–Hausdorff distància entre esferes de diferents dimensions.

Keywords: *Hausdorff, metric, correspondance.*

Abstract

The Gromov–Hausdorff distance between metric spaces X and Y, denoted by $d_{GH}(X,Y)$, quantifies the extent to which X and Y fail to be isometric. The Gromov–Hausdorff distance is used in many areas of geometry, in applications to shape and data comparison/classification, one aims to estimate either the Gromov–Hausdorff distance between spaces or the Gromov–Wasserstein distance, which is one of its optimal transport induced variants.

Let A, B be pseudo-metric spaces. The *Gromov–Hausdorff distance* (see [2]) between A and B, denoted by $d_{GH}(A,B)$, is the infimum of all $\varepsilon \geq 0$ so that there is a pseudo-metric space M and isometric embeddings $i_A\colon A \to M$ and $i_B\colon B \to M$ such that $d_M(i_A(A), i_B(B)) \leq \varepsilon$, where d_M denotes Hausdorff distance in M. Then we prove that we can actually restrict ourselves to pseudo-metrics on the disjoint union of A and B.

We introduce correspondences between sets and the concept of distortion of a correspondence in order to prove that the Gromov–Hausdorff distance can be computed using them. For any two pseudo-metric spaces X and Y,

$$d_{GH}(X,Y) = \frac{1}{2}\inf_C\{\mathrm{dis}(C)\},$$

where the infimum is taken over all correspondences C between X and Y. The set of isometry classes of compact metric spaces endowed with the Gromov–Hausdorff distance is a metric space.

We study the structure of the metric space of metrics on a given set. We focus on the case where the given space is a complete and compact metric space. Then we study the set of closed relations and the subset of closed correspondences (see [3]), which turns out to be a compact set. We prove that the distortion function is a continuous function. Hence we obtain the following result: For any two compact metric spaces X and Y there exists a correspondence R such that $d_{GH}(X,Y) = \frac{1}{2}\,\mathrm{dis}(R)$.

We focus on the case of estimating Gromov–Hausdorff distances between spheres of different dimensions (see [1, 5], for a generalization see [4]). We relate Gromov–Hausdorff distance, Borsuk–Ulam theorems, and Vietoris–Rips complexes as follows. Estimating the Gromov–Haudorff distance $d_{GH}(X,Y)$ for metric spaces X and Y involves bounding the distortion of a function $f\colon X \to Y$, which measures the extent to which f fails to preserve distances; the more functions between X and Y distort the metrics, the larger $d_{GH}(X,Y)$ must be. When X and Y are spheres, it is sufficient to consider odd functions. We transform an odd function $f : \mathbb{S}^k \to \mathbb{S}^n$ into a continuous odd map between Vietoris–Rips complexes. Then we obstruct the existence of such maps with the $\mathbb{Z}/2$ equivariant topology of Vietoris–Rips complexes, measured via the following quantity: For $k \geq n$, we define

$$c_{n,k} = \inf\{r \geq 0 \mid \text{there exists an odd map } \mathbb{S}^k \to VR(\mathbb{S}^n; r)\}.$$

Due to a theorem of Hausmann, there is a homotopy equivalence $VR(\mathbb{S}^n; r) \simeq \mathbb{S}^n$ for sufficiently small r, and moreover there is an odd map $f\colon VR(\mathbb{S}^n; r) \to \mathbb{S}^n$. The Borsuk–Ulam theorem then implies that no odd map $\mathbb{S}^k \to VR(\mathbb{S}^n; r)$ exists for such r unless $k \leq n$. In particular, $c_{n,n} = 0$. Therefore, the quantity $c_{n,k}$ represents the amount by which $\mathbb{S}^n$ needs to be "thickened" until it admits an odd map from $\mathbb{S}^k$.

We find bounds for the Gromov–Hausdorff distance between spheres: For all $k \geq n$, the following inequalities hold:

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) \geq \inf\{\mathrm{dis}(f) \mid f\colon \mathbb{S}^k \to \mathbb{S}^n \text{ is odd}\} \geq c_{n,k}.$$

And that for every $n \geq 1$, we have that $d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1}) \leq \pi/3$.

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